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Nyström method for Cauchy singular integral equations with negative index ☆

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ABSTRACT

In this paper, the authors propose a Nyström method to approximate the solutions of Cauchy singular integral equations with constant coefficients having a negative index. They consider the equations in spaces of continuous functions with weighted uniform norm. They prove the stability and the convergence of the method and show some numerical tests that confirm the error estimates.

1. Introduction

Let us consider the following Cauchy singular integral equations with constant coefficients

$$af(y)v^{\alpha,\beta}(y) + \frac{b}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}v^{\alpha,\beta}(x)\mathrm{d}x + \mu\int_{-1}^{1}k(x,y)f(x)v^{\alpha,\beta}(x)\mathrm{d}x = g(y), \tag{1}$$

where $|y| < 1$, $a, b \in \mathbb{R}$ are constants such that $a^2 + b^2 = 1$, $b \neq 0$, $\mu \in \mathbb{R}$ and k and g are given functions on $(-1, 1)^2$ and $(-1, 1)$, respectively. The function f is the unknown of the equation and $v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ is a Jacobi weight whose exponents $-1 < \alpha, \beta < 1$ are given by

$$\alpha = M - \frac{1}{2\pi \mathrm{i}}\log\left(\frac{a+\mathrm{i}b}{a-\mathrm{i}b}\right), \qquad \beta = N + \frac{1}{2\pi \mathrm{i}}\log\left(\frac{a+\mathrm{i}b}{a-\mathrm{i}b}\right),$$

with M and N integers chosen so that $\chi = -(\alpha+\beta) = -(M+N)$ is restricted to $\chi \in \{-1, 0, 1\}$.

This kind of integral equation appears in several problems of applied sciences and a wide literature on this topic is available. Among them, we mention the fundamental books and papers [1–12] and the references therein. Recently, in [13], such class of equations with index $\chi \in \{0, 1\}$ were considered in spaces of continuous functions with uniform norm. Using the regularization method [8,10,11], Fredholm equations are obtained and they are solved using projection-type methods.

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In this paper, we consider equations of the form (1) with index $\chi = -1$ in spaces of continuous functions with uniform norm. The numerical treatment of Cauchy singular integral equations in case of negative index has not been so widely developed in literature ([14,6] see also [5,12]). The linear system derived from direct methods (for instance quadrature, collocation or discrete collocation) is overdetermined and, then, it becomes necessary to exclude one of the equations. However, the criteria used in choosing such an equation are not clear.

In the sequel, in order to fix the ideas, we shall consider Cauchy singular integral equations of the following type

$$(Df)(y) + \mu(Kf)(y) = g(y), \quad |y| < 1, \tag{2}$$

where $\mu \in \mathbb{R}$ and the operators D and K are defined as

$$(Df)(y) = \cos(\pi\alpha)f(y)v^{\alpha,1-\alpha}(y) - \frac{\sin(\pi\alpha)}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}v^{\alpha,1-\alpha}(x)\mathrm{d}x, \tag{3}$$

$$(Kf)(y) = \int_{-1}^{1} k(x,y)f(x)v^{\alpha,1-\alpha}(x)\mathrm{d}x, \tag{4}$$

respectively, with $\alpha \in (0, 1)$. In other words, we take the constant coefficients a and b appearing in (1) as $a = \cos(\pi\alpha)$ and $b = -\sin(\pi\alpha)$, respectively. The other possible choice is $a = -\cos\pi\alpha$, $b = \sin\pi\alpha$ and, in this case, what follows can be repeated word by word.

Following [13,15], the procedure we propose here consists in reducing (1), under suitable assumptions on k and g, to an equivalent regularized Fredholm integral equation and in solving the latter by a Nyström-type method. This approach, based on some mapping properties of the dominant operator D related to Eq. (1) (see Section 2.2), permits the solution of a determined and well conditioned linear system.

This paper is organized as follows. In Section 2 we introduce some function spaces and show some mapping properties of the operators D and K. In Section 3 we describe the numerical method, and in Section 4 we focus on the computational aspects. In Section 5 we prove the results stated in Sections 3 and 4. Finally, in Section 6 we give some numerical tests.

2. Preliminaries

In this paper, $\mathcal{C}$ will denote a positive constant which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$. If $A, B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B, such that $\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B$.

2.1. Functional spaces

We are going to consider the integral equations (2) in the space

$$C_v = \left\{ f \in C^0((-1,1)) : \lim_{|x|\to 1}(fv)(x) = 0 \right\},$$

where $C^0(A)$ is the collection of the continuous functions in $A \subset [-1, 1]$ and $v(x) := v^{\rho,\theta}(x) = (1-x)^{\rho}(1+x)^{\theta}$, $\rho, \theta \geq 0$, is a Jacobi weight.

In case $\rho = 0$ (respectively, $\theta = 0$) C_v consists of all continuous functions on $(-1, 1]$ (respectively, $[-1, 1)$) such that

$$\lim_{x\to -1}(fv)(x) = 0 \quad \left(\text{respectively, } \lim_{x\to 1}(fv)(x) = 0\right).$$

In the case where $\rho = \theta = 0$, we set $C_v = C^0([-1, 1])$. The space C_v equipped with the norm

$$\|f\|_{C_v} := \|fv\| := \max_{|x|\leq 1}|(fv)(x)|$$

is complete. Sometimes, for brevity of notations, we shall write $\|f\|_A := \max_{x\in A}|f(x)|$, for $A \subseteq [-1, 1]$.

In the sequel we will also consider functions belonging to the Zygmund space $Z_r(v)$ defined as follows

$$Z_r(v) = \left\{ f \in C_v : \sup_{t>0}\frac{\Omega_\varphi^k(f,t)_v}{t^r} < +\infty,\ k > r > 0 \right\}, \quad r \in \mathbb{R}^+, k \in \mathbb{N},$$

by means of the main part of the modulus of continuity [16]

$$\Omega_\varphi^k(f,t)_v = \sup_{0<h\leq t}\|(\Delta_{h\varphi}^k f)v\|_{I_{h,k}},$$

where $I_{h,k} = [-1 + 4k^2h^2, 1 - 4k^2h^2]$, $0 < t < 1$, $\varphi(x) = \sqrt{1-x^2}$ and

$$\Delta_{h\varphi}^{k} f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + h\varphi(x)\left(\frac{k}{2} - i\right)\right).$$

We equip $Z_r(v)$ with the following norm

$$\|f\|_{Z_r(v)} = \|fv\| + \sup_{t>0} \frac{\Omega_{\varphi}^{k}(f,t)_v}{t^r}.$$

For brevity, we will set $Z_r := Z_r(v^{0,0})$.

2.2. Mapping properties of D and K

The dominant operator D defined in (3) and the operator

$$(\widehat{D}f)(y) = \cos(\pi\alpha) f(y) v^{-\alpha,\alpha-1}(y) + \frac{\sin(\pi\alpha)}{\pi} \int_{-1}^{1} \frac{f(x)}{x-y} v^{-\alpha,\alpha-1}(x)\mathrm{d}x, \tag{5}$$

with $\alpha \in (0, 1)$, satisfy the following theorem ([17], pp. 396–397).

Theorem 2.1. *For $0 < \alpha < 1$ and $r > 0$, the operators $D : Z_r(v^{\alpha,1-\alpha}) \to Z_r$ and $\widehat{D} : Z_r \to Z_r(v^{\alpha,1-\alpha})$ are continuous linear maps. For all $f \in Z_r(v^{\alpha,1-\alpha})$ we have*

$$\widehat{D}Df = f, \tag{6}$$

i.e. D is a right inverse of $\widehat{D}$ in the pair $(Z_r, Z_r(v^{\alpha,1-\alpha}))$, and for all $f \in Z_r$

$$D\widehat{D}f = f - \frac{\int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x)\mathrm{d}x}{\int_{-1}^{1} v^{-\alpha,\alpha-1}(x)\mathrm{d}x}. \tag{7}$$

Moreover, there holds

$$\|\widehat{D}f\|_{Z_r(v^{\alpha,1-\alpha})} \sim \|f\|_{Z_r}, \tag{8}$$

where the constants in "$\sim$" are independent of f.

Furthermore, denoting by $\{p_m^{\rho,\theta}\}_{m\in\mathbb{N}}$ the sequence of the polynomials which are orthonormal with respect to the Jacobi weight $v^{\rho,\theta}$, the following property of the operator $\widehat{D}$ is well known (see, for instance, [12])

$$\widehat{D}p_m^{-\alpha,\alpha-1} = p_{m-1}^{\alpha,1-\alpha}, \qquad (p_{-1}^{\alpha,1-\alpha} = 0), \quad m = 0, 1, 2, \ldots. \tag{9}$$

Setting $k_x(y) = k_y(x) = k(x, y)$, one can prove that the operator K, defined by (4), satisfies the following theorem.

Theorem 2.2. *Let $0 < \alpha < 1$ and let $v^{\rho,\theta}$ be a Jacobi weight such that*

$$0 \le \rho < \alpha + 1, \quad 0 \le \theta < 2 - \alpha. \tag{10}$$

If $r > 0$ and the kernel $k(x, y)$ verifies the condition

$$\sup_{|x|\le 1} \|k_x\|_{Z_r(v^{\rho,\theta})} < +\infty \tag{11}$$

then $K : C_{v^{\rho,\theta}} \to Z_r(v^{\rho,\theta})$ is a bounded linear operator. Moreover $K : C_{v^{\rho,\theta}} \to C_{v^{\rho,\theta}}$ is compact.

3. Numerical method

In the present section, we want to describe a numerical procedure to approximate the solution of the integral equation (2).

Assuming that, for some $r > 0$, $k_x \in Z_r$, uniformly with respect to x, and $g \in Z_r$, we apply the operator $\widehat{D}$ defined by (5) to the left side of (2). By (6) we get

$$f(y) + \mu \int_{-1}^{1} (\widehat{D}k_x)(y) f(x) v^{\alpha,1-\alpha}(x)\mathrm{d}x = (\widehat{D}g)(y), \quad |y| < 1 \tag{12}$$

which is a Fredholm integral equation of the second kind. We note that, in general, (12) is not equivalent to (2), but, taking into account (7), a sufficient condition for the equivalence of the two problems is [18,11]

$$\int_{-1}^{1} k(x,y)v^{-\alpha,\alpha-1}(x)\mathrm{d}x = 0 = \int_{-1}^{1} g(x)v^{-\alpha,\alpha-1}(x)\mathrm{d}x. \tag{13}$$

Thus, under our assumptions on k and g, if f is a solution of (12) then f is a solution of (2), (13), too. Using the notations

$$\Phi(x,y) = (\widehat{D}k_x)(y) \quad \text{and} \quad G(y) = (\widehat{D}g)(y),$$

(12) can be rewritten as

$$f(y) + \mu\int_{-1}^{1} \Phi(x,y)f(x)v^{\alpha,1-\alpha}(x)\mathrm{d}x = G(y), \quad |y| < 1. \tag{14}$$

We are going to consider (14) in the space $C_{v^{\alpha+\gamma,1-\alpha+\delta}}$ with

$$0 \le \gamma < 1-\alpha, \quad 0 \le \delta < \alpha. \tag{15}$$

In order to solve (14) by means of a Nyström-type method, we approximate the integral appearing in it by using a suitable Gaussian rule and, then, we get the approximating equations

$$f_m(y) + \mu\sum_{k=1}^{m} \lambda_{m,k}^{\alpha,1-\alpha}\Phi(x_k,y)f_m(x_k) = G(y), \quad m = 1,2,\ldots, \tag{16}$$

where $x_k = x_{m,k}^{\alpha,1-\alpha}$, $k = 1,\ldots,m$, denote the zeros of the polynomial $p_m^{\alpha,1-\alpha}$ and $\lambda_{m,k}^{\alpha,1-\alpha}$, $k = 1,\ldots,m$, the corresponding Christoffel numbers.

Multiplying both sides of (16) by $v^{\alpha+\gamma,1-\alpha+\delta}$ and collocating on the quadrature knots, we obtain the linear system

$$\sum_{k=1}^{m}\left[\delta_{i,k} + \mu\lambda_{m,k}^{\alpha,1-\alpha}\frac{v^{\alpha+\gamma,1-\alpha+\delta}(x_i)}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)}\Phi(x_k,x_i)\right]a_k = G(x_i)v^{\alpha+\gamma,1-\alpha+\delta}(x_i), \quad i = 1,2,\ldots,m, \tag{17}$$

where $a_k = f_m(x_k)v^{\alpha+\gamma,1-\alpha+\delta}(x_k)$, $k = 1,2,\ldots,m$, are the unknowns.

If $(a_1^*,\ldots,a_m^*)$ is the unique solution of system (17), then we construct the weighted Nyström interpolating function

$$v^{\alpha+\gamma,1-\alpha+\delta}(y)f_m^*(y) = v^{\alpha+\gamma,1-\alpha+\delta}(y)\left[G(y) - \mu\sum_{k=1}^{m}\frac{\lambda_{m,k}^{\alpha,1-\alpha}}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)}\Phi(x_k,y)a_k^*\right]. \tag{18}$$

In the next theorem we will show that, if the original problem (2) is unisolvent and (13) is fulfilled, for a sufficiently large m, the linear system (17) is also uniquely solvable as well as well conditioned. Moreover, the corresponding approximate solution f_m^*, defined by (18), converges to the exact one in the space $C_{v^{\alpha+\gamma,1-\alpha+\delta}}$.

Before stating it, we premise the following crucial lemma.

Lemma 3.1. *If* $0 < \alpha < 1$ *and* γ, δ *satisfy* (15)*, then we have*

$$\|G\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})} \le \mathcal{C}\|g\|_{Z_r}, \tag{19}$$

$$\sup_{|x|\le 1}\|\Phi_x\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})} \le \mathcal{C}\|k_x\|_{Z_r}, \tag{20}$$

$$\sup_{|y|\le 1} v^{\alpha+\gamma,1-\alpha+\delta}(y)\|\Phi_y\|_{Z_r} \le \mathcal{C}\left(\sup_{|y|\le 1} v^{\gamma,\delta}(y)\|k_y\|_{Z_r} + \sup_{|y|\le 1} v^{\gamma,\delta}(y)\left\|\frac{\partial}{\partial y}k_y\right\|_{Z_r}\right), \tag{21}$$

with $r > 0$, $\Phi_x(y) = \Phi_y(x) = \Phi(x,y)$ *and* $\mathcal{C} \ne \mathcal{C}(g,k,x,y)$.

Now, denoting by A_m the matrix of the coefficients of system (17) and by $\mathrm{cond}(A_m)$ its condition number in uniform norm, the following theorem holds true

Theorem 3.1. *Let*

$$\|g\|_{Z_r} < +\infty, \tag{22}$$

$$\sup_{|x|\le 1}\|k_x\|_{Z_r} < +\infty, \tag{23}$$

$$\sup_{|y|\le 1} v^{\gamma,\delta}(y)\|k_y\|_{Z_r} + \sup_{|y|\le 1} v^{\gamma,\delta}(y)\left\|\frac{\partial}{\partial y}k_y\right\|_{Z_r} < +\infty \tag{24}$$

hold true and condition (13) be fulfilled. Assuming that Eq. (2) is uniquely solvable in $C_{v^{\alpha+\gamma,1-\alpha+\delta}}$ with γ, δ as in (15), for any fixed g, for a sufficiently large m (say $m > m_0$), the matrix A_m is invertible and satisfies

$$\sup_m \operatorname{cond}(A_m) < +\infty. \tag{25}$$

Moreover, the Nyström interpolating function f_m^* defined by (18) converges to the unique solution f^* of (14) and

$$\|(f^* - f_m^*)v^{\alpha+\gamma,1-\alpha+\delta}\| \le \frac{\mathcal{C}}{m^r}\|f^*\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}, \tag{26}$$

where $\mathcal{C} \ne \mathcal{C}(m, f^*)$.

4. Computational aspects

In this section, we focus on some problems which can arise in the application of the above described method. We premise some notations.

In the sequel $L_m^{\rho,\theta}$ will stand for the Lagrange interpolation operator based on the zeros $x_{m,k}^{\rho,\theta}$, $k = 1, \ldots, m$, of the orthonormal polynomial $p_m^{\rho,\theta}$, i.e.

$$L_m^{\rho,\theta}(f, x) = \sum_{k=1}^{m} f(x_{m,k}^{\rho,\theta}) l_{m,k}^{\rho,\theta}(x)$$

where

$$l_{m,k}^{\rho,\theta}(x) = \frac{p_m^{\rho,\theta}(x)}{[p_{m+1}^{\rho,\theta}]'(x_{m,k}^{\rho,\theta})(x - x_{m,k}^{\rho,\theta})}$$

is the k-th fundamental Lagrange polynomial.

Moreover, we will denote by $t_j = x_{m+1,j}^{-\alpha,\alpha-1}$, $j = 1, \ldots, m+1$, the zeros of $p_m^{-\alpha,\alpha-1}$ and by $\lambda_{m+1,j}^{-\alpha,\alpha-1}$, $j = 1, \ldots, m+1$, the corresponding Christoffel numbers.

According to (17) and (18), in order to construct the matrix A_m and the right-hand side vector of the linear system as well as the approximate solution f_m^*, one has to compute the quantities

$$\Phi(x_k, y) = (\widehat{D}k_{x_k})(y), \quad k = 1, \ldots, m,$$

and

$$G(y) = (\widehat{D}g)(y),$$

for $|y| < 1$. When their analytical expressions are not available, we suggest to approximate them by

$$\Phi_{m+1}(x_k, y) = (\widehat{D}L_{m+1}^{-\alpha,\alpha-1}k_{x_k})(y)$$

and

$$G_{m+1}(y) = (\widehat{D}L_{m+1}^{-\alpha,\alpha-1}g)(y),$$

respectively.

In virtue of the property (9) of the operator $\widehat{D}$, one can easily express $\Phi_{m+1}(x_k, y)$ and $G_{m+1}(y)$, for $y \in (-1, 1)$, as follows

$$\Phi_{m+1}(x_k, y) = \sum_{j=1}^{m+1} k(x_k, t_j)\lambda_{m+1,j}^{-\alpha,\alpha-1} \sum_{i=0}^{m} p_i^{-\alpha,\alpha-1}(t_j)p_{i-1}^{\alpha,1-\alpha}(y)$$

$$G_{m+1}(y) = \sum_{j=1}^{m+1} g(t_j)\lambda_{m+1,j}^{-\alpha,\alpha-1} \sum_{i=0}^{m} p_i^{-\alpha,\alpha-1}(t_j)p_{i-1}^{\alpha,1-\alpha}(y).$$

Nevertheless, taking into account that

$$\widehat{D}\left[\frac{p_{m+1}^{-\alpha,\alpha-1}}{\cdot - t_j}\right](y) = \frac{p_m^{\alpha,1-\alpha}(y) - p_m^{\alpha,1-\alpha}(t_j)}{y - t_j}$$

and

$$\frac{p_m^{\alpha,1-\alpha}(t_j)}{[p_{m+1}^{-\alpha,\alpha-1}]'(t_j)} = \frac{\sin(\pi\alpha)}{\pi}\lambda_{m+1,j}^{-\alpha,\alpha-1},$$

when $y = x_i$, $i = 1, \dots, m$, one can also obtain

$$\Phi_{m+1}(x_k, x_i) = \frac{\sin(\pi\alpha)}{\pi} \sum_{j=1}^{m+1} \frac{k(x_k, t_j)}{t_j - x_i} \lambda_{m+1,j}^{-\alpha,\alpha-1}, \tag{27}$$

$$G_{m+1}(x_i) = \frac{\sin(\pi\alpha)}{\pi} \sum_{j=1}^{m+1} \frac{g(t_j)}{t_j - x_i} \lambda_{m+1,j}^{-\alpha,\alpha-1}. \tag{28}$$

Let us remark that, in (27) and (28) the distance between the zeros x_i and t_j, $\forall\, i = 1, \dots, m$, $\forall\, j = 1, \dots, m+1$, is large enough to avoid the numerical cancellation. In fact, letting $x_i = \cos\theta_{m,i}^{\alpha,1-\alpha}$ and $t_j = \cos\theta_{m+1,j}^{-\alpha,\alpha-1}$,

$$\min_{i,j} |\theta_{m,i}^{\alpha,1-\alpha} - \theta_{m,j}^{-\alpha,\alpha-1}| \geq \frac{\mathcal{C}}{m} \tag{29}$$

holds [19]. Then we replace system (17) by

$$\sum_{k=1}^{m} \left[\delta_{i,k} + \mu\, \lambda_{m,k}^{\alpha,1-\alpha} \frac{v^{\alpha+\gamma,1-\alpha+\delta}(x_i)}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)} \Phi_{m+1}(x_k, x_i) \right] \bar{a}_k = G_{m+1}(x_i) v^{\alpha+\gamma,1-\alpha+\delta}(x_i), \quad i = 1, 2, \dots, m \tag{30}$$

and, in correspondence of a solution $(\bar{a}_1^*, \dots, \bar{a}_m^*)$ of (30), we construct the approximating function

$$v^{\alpha+\gamma,1-\alpha+\delta}(y) f_m^{**}(y) = v^{\alpha+\gamma,1-\alpha+\delta}(y) \left[G_{m+1}(y) - \mu \sum_{k=1}^{m} \frac{\lambda_{m,k}^{\alpha,1-\alpha}}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)} \Phi_{m+1}(x_k, y) \bar{a}_k^* \right]. \tag{31}$$

In the following proposition we establish that system (30) is unisolvent as well as (17) and estimate the new error $\|(f^* - f_m^{**}) v^{\alpha+\gamma,1-\alpha+\delta}\|$.

Proposition 4.1. *Under the hypotheses of Theorem 3.1, for a sufficiently large m (say $m > m_0$), the matrix A_m^* of the coefficients of system (30) is invertible and*

$$\lim_m \frac{\operatorname{cond}(A_m^*)}{\operatorname{cond}(A_m)} = 1, \tag{32}$$

Moreover, if

$$\max\left\{0, -\frac{\alpha}{2} + \frac{1}{4}\right\} \leq \gamma < \min\left\{-\frac{\alpha}{2} + \frac{3}{4}, 1 - \alpha\right\} \tag{33}$$
$$\max\left\{0, \frac{\alpha}{2} - \frac{1}{4}\right\} \leq \delta < \min\left\{\frac{\alpha}{2} + \frac{1}{4}, \alpha\right\},$$

the approximate solution f_m^{**} defined by (31) converges to the unique solution f^* of (14) and satisfies the estimate

$$\|(f^* - f_m^{**}) v^{\alpha+\gamma,1-\alpha+\delta}\| = \mathcal{O}\left(\frac{\log^2 m}{m^r}\right), \tag{34}$$

where the constant in $\mathcal{O}$ is independent of m.

Therefore, the condition numbers of systems (17) and (30) are comparable and, if system (30) replaces (17), then the estimate (26) is perturbed by a $\log^2 m$ factor.

5. Proofs

We first give some notations and preliminary results.
For any linear operator $T : C_{v^{\rho,\theta}} \to C_{v^{\rho,\theta}}$, let $\|T\|_{C_{v^{\rho,\theta}} \to C_{v^{\rho,\theta}}}$ be the norm of T, i.e.

$$\|T\|_{C_{v^{\rho,\theta}} \to C_{v^{\rho,\theta}}} = \sup_{\|f\|_{C_{v^{\rho,\theta}}} = 1} \|Tf\|_{C_{v^{\rho,\theta}}}.$$

If $\mathbb{P}_m$ is the set of all algebraic polynomials of degree at most m, we denote by

$$E_m(f)_{v^{\rho,\theta}} = \inf_{P \in \mathbb{P}_m} \|(f - P) v^{\rho,\theta}\|$$

the error of best weighted approximation of a function f in $C_{v^{\rho,\theta}}$ by means of polynomials in $\mathbb{P}_m$. For brevity, we will set $E_m(f) := E_m(f)_{v^{0,0}}$. For all functions $f \in C_{v^{\rho,\theta}}$ we have [16, p. 94]

$$E_m(f)_{v^{\rho,\theta}} \le \mathcal{C} \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(f,t)_{v^{\rho,\theta}}}{t} \mathrm{d}t. \tag{35}$$

In particular, from (35) we deduce

$$E_m(f)_{v^{\rho,\theta}} \le \frac{\mathcal{C}}{m^r} \|f\|_{Z_r(v^{\rho,\theta})}, \quad \forall f \in Z_r(v^{\rho,\theta}). \tag{36}$$

We define in $C_{v^{\rho,\theta}}$ the r-th φ-modulus of continuity as follows

$$\omega_\varphi^k(f,t)_{v^{\rho,\theta}} = \Omega_\varphi^k(f,t)_{v^{\rho,\theta}} + \inf_{P\in\mathbb{P}_{k-1}} \|(f-P)v^{\rho,\theta}\|_{C([-1,-1+4k^2t^2])} + \inf_{P\in\mathbb{P}_{k-1}} \|(f-P)v^{\rho,\theta}\|_{C([1-4k^2t^2,1])},$$

where $0 < t \le \frac{1}{2k}$. We will also use the notation $\omega_\varphi = \omega_\varphi^1$.

Note that, if $f \in Z_r(v^{\rho,\theta})$, $r \in \mathbb{R}^+$, then $\omega_\varphi^k(f,t)_{v^{\rho,\theta}} \sim \Omega_\varphi^k(f,t)_{v^{\rho,\theta}}$ [16, p. 94].

Proof of Theorem 2.2. At first we want to prove that for any $f \in C_{v^{\rho,\theta}}$ one has

$$\|Kf\|_{Z_r(v^{\rho,\theta})} \le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}}, \quad \mathcal{C} \ne \mathcal{C}(f). \tag{37}$$

For $k > 0$ and $h \le t$, t "small" (say $t < 1$), we have

$$\begin{aligned} v^{\rho,\theta}(y)|\Delta_{h\varphi}^k(Kf)(y)| &= v^{\rho,\theta}(y)\left|\int_{-1}^1 (\Delta_{h\varphi}^k k_x)(y)(fv^{\rho,\theta})(x)\frac{v^{\alpha,1-\alpha}(x)}{v^{\rho,\theta}(x)}\mathrm{d}x\right| \\ &\le \|f\|_{C_{v^{\rho,\theta}}} \int_{-1}^1 |v^{\rho,\theta}(y)(\Delta_{h\varphi}^k k_x)(y)| v^{\alpha-\rho,1-\alpha-\theta}(x)\mathrm{d}x. \end{aligned}$$

Taking the supremum on $y \in I_{h,k} = [-1+4k^2h^2, 1-4k^2h^2]$ first and then the supremum on $0 < h \le t$, we get

$$\begin{aligned} \Omega_\varphi^k(Kf,t)_{v^{\rho,\theta}} &\le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}} \int_{-1}^1 \Omega_\varphi^k(k_x,t)_{v^{\rho,\theta}} v^{\alpha-\rho,1-\alpha-\theta}(x)\mathrm{d}x \\ &\le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}} \sup_{|x|\le 1} \Omega_\varphi^k(k_x,t)_{v^{\rho,\theta}} \int_{-1}^1 v^{\alpha-\rho,1-\alpha-\theta}(x)\mathrm{d}x. \end{aligned}$$

Thus, by (10) and (11), we have

$$\Omega_\varphi^k(Kf,t)_{v^{\rho,\theta}} \le \mathcal{C}\|f\|_{C_{v^{\rho,\theta}}} \sup_{|x|\le 1} \Omega_\varphi^k(k_x,t)_{v^{\rho,\theta}} \le \mathcal{C}t^r\|f\|_{C_{v^{\rho,\theta}}},$$

from which we deduce that $Kf \in Z_r(v^{\rho,\theta})$, i.e. (37). Moreover, recalling (35), we obtain

$$E_m(Kf)_{v^{\rho,\theta}} \le \frac{\mathcal{C}}{m^r}\|f\|_{C_{v^{\rho,\theta}}}$$

and then

$$\lim_m \left(\sup_{\|f\|_{v^{\rho,\theta}}=1} E_m(Kf)_{v^{\rho,\theta}} \right) = 0,$$

i.e. $K : C_{v^{\rho,\theta}} \to C_{v^{\rho,\theta}}$ is compact (see [20, p. 44]). □

In order to prove Lemma 3.1 we need the following proposition.

Proposition 5.1. *Let $0 < \alpha < 1$. Then, for $f \in Z_r(v^{\gamma,\delta})$ and γ, δ satisfying (15), we have*

$$|v^{\alpha+\gamma,1-\alpha+\delta}(y)(\widehat{D}f)(y)| \le \mathcal{C}\left[\|fv^{\gamma,\delta}\| + \int_0^1 \frac{\omega_\varphi(f,t)_{v^{\gamma,\delta}}}{t}\mathrm{d}t\right], \tag{38}$$

where $|y| \le 1$ and $\mathcal{C} \ne \mathcal{C}(f,y)$.

Proof. It is not hard to deduce (38) by [21, Theorem 3.1]. □

Proof of Lemma 3.1. Recalling (8), we obtain

$$\|G\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}=\|\widehat{D}g\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}\le\|\widehat{D}g\|_{Z_r(v^{\alpha,1-\alpha})}\le\mathcal{C}\|g\|_{Z_r},$$

i.e. (19). Analogously it is possible to prove (20).

It remains to prove (21). We have

$$\begin{aligned}v^{\alpha+\gamma,1-\alpha+\delta}(y)\|\Phi_y\|_{Z_r}&=v^{\alpha+\gamma,1-\alpha+\delta}(y)\|\Phi_y\|+v^{\alpha+\gamma,1-\alpha+\delta}(y)\sup_{t>0}\frac{\Omega_\varphi^k(\Phi_y,t)}{t^r}\\&=:B_1+B_2.\end{aligned}\tag{39}$$

With regards to B_1, applying (38) we get

$$\begin{aligned}v^{\alpha+\gamma,1-\alpha+\delta}(y)|\Phi(x,y)|&=v^{\alpha+\gamma,1-\alpha+\delta}(y)|(\widehat{D}k_x)(y)|\\&\le\mathcal{C}\left[\sup_{|y|\le1}v^{\gamma,\delta}(y)|k(x,y)|+\int_0^1\frac{\omega_\varphi(k_x,t)_{v^{\gamma,\delta}}}{t}\mathrm{d}t\right].\end{aligned}$$

Since [16, p. 92]

$$\omega_\varphi(k_x,t)_{v^{\gamma,\delta}}\le\mathcal{C}t\sup_{|y|\le1}\left|\varphi(y)v^{\gamma,\delta}(y)\frac{\partial}{\partial y}k(x,y)\right|,$$

we obtain

$$B_1\le\mathcal{C}\sup_{|y|\le1}v^{\gamma,\delta}(y)\sup_{|x|\le1}|k(x,y)|+\mathcal{C}\sup_{|y|\le1}v^{\gamma,\delta}(y)\sup_{|x|\le1}\left|\frac{\partial}{\partial y}k(x,y)\right|.\tag{40}$$

Analogously we deduce

$$\begin{aligned}v^{\alpha+\gamma,1-\alpha+\delta}(y)|\Delta^k_{h\varphi(x)}\Phi(x,y)|&=v^{\alpha+\gamma,1-\alpha+\delta}(y)|\widehat{D}(\Delta^k_{h\varphi(x)}k_x)(y)|\\&\le\mathcal{C}\sup_{|y|\le1}v^{\gamma,\delta}(y)|\Delta^k_{h\varphi(x)}k(x,y)|+\mathcal{C}\sup_{|y|\le1}v^{\gamma,\delta}(y)\left|\Delta^k_{h\varphi(x)}\frac{\partial}{\partial y}k(x,y)\right|\end{aligned}$$

and, taking the supremum on $x\in[-1+4k^2h^2,1-4k^2h^2]$ first and then the supremum on $0<h\le t$, we get

$$B_2\le\mathcal{C}\sup_{|y|\le1}v^{\gamma,\delta}(y)\sup_{t>0}\frac{\Omega_\varphi^k(k_y,t)}{t^r}+\mathcal{C}\sup_{|y|\le1}v^{\gamma,\delta}(y)\sup_{t>0}\frac{\Omega_\varphi^k\left(\frac{\partial}{\partial y}k_y,t\right)}{t^r}.\tag{41}$$

Finally, combining (40) and (41) with (39), (21) follows. □

Proof of Theorem 3.1. For brevity of notations, we set $u=v^{\alpha+\gamma,1-\alpha+\delta}$. Since, under the assumptions (23) and (15), (20) holds, from Theorem 2.2, applied with $\rho=\alpha+\gamma$ and $\theta=1-\alpha+\delta$, we deduce that the operator

$$(\bar{K}f)(y)=\int_{-1}^1\Phi(x,y)f(x)v^{\alpha,1-\alpha}(x)\mathrm{d}x$$

is compact as a map from C_u into itself and that $\bar{K}f\in Z_r(u)$ for all $f\in C_u$.

Now we proceed to the proof of (26). In order to apply [22, Theorem 4.1.1. p.106], letting

$$(\bar{K}_mf)(y)=\sum_{i=1}^m\Phi(x_i,y)f(x_i)\lambda_i^{\alpha,1-\alpha},$$

we prove that

$$\|\bar{K}_mf\|_{C_u}\le\mathcal{C}\|f\|_{C_u},\quad\mathcal{C}\ne\mathcal{C}(m,f),\tag{42}$$

$$\lim_m\|(\bar{K}-\bar{K}_m)f\|_{C_u}=0,\quad\forall f\in C_u,\tag{43}$$

and

$$\lim_m\|(\bar{K}-\bar{K}_m)\bar{K}_m\|_{C_u\to C_u}=0.\tag{44}$$

We have

$$u(y)|(\bar{K}_m f)(y)| \le \sum_{i=1}^{m} u(y)|\Phi_{x_i}(y)||(fu)(x_i)|\frac{\lambda_{m,i}^{\alpha,1-\alpha}}{u(x_i)} \le \mathcal{C}\|f\|_{C_u} \sup_{|x|\le 1}\|\Phi_x u\| \sum_{i=1}^{m}\frac{\lambda_{m,i}^{\alpha,1-\alpha}}{u(x_i)}.$$

Since (see [23])

$$(1\pm x)\sim(1\pm x_i),\quad \forall x\in[x_i,x_{i+1}],$$

and

$$\lambda_{m,i}^{\alpha,1-\alpha}\sim \Delta x_i v^{\alpha,1-\alpha}(x_i),\quad i=1,\dots,m,$$

with $\Delta x_i = x_{i+1}-x_i$ ($x_{m+1}=1$) and the constants in "$\sim$" independent of m and i, one has

$$\sum_{i=1}^{m}\frac{\lambda_{m,i}^{\alpha,1-\alpha}}{u(x_i)} \le \mathcal{C}\int_{-1}^{1} v^{-\gamma,-\delta}(x)\mathrm{d}x \le \mathcal{C} \tag{45}$$

by (15). Therefore, by (20), (23) and (45), (42) follows.

Now we prove (43). By standard computations, it is possible to deduce that

$$|[(\bar{K}-\bar{K}_m)f](y)u(y)| \le \mathcal{C}u(y)E_{2m-2}(\Phi_y f)_u \le \mathcal{C}\|f\|_{C_u} u(y)E_{m-1}(\Phi_y) + 2u(y)\|\Phi_y\|E_{m-1}(f)_u. \tag{46}$$

Thus, using (36) together with (21) and (24), we obtain

$$|[(\bar{K}-\bar{K}_m)f](y)u(y)| \le \frac{\mathcal{C}}{m^r}\|f\|_{C_u} + \mathcal{C}E_{m-1}(f)_u \tag{47}$$

and, $\forall f\in C_u$, (43) follows.

In order to prove (44) we first write

$$u(y)|\Delta_{h\varphi}^k(\bar{K}_m f)(y)| = \sum_{i=1}^{m} |u(y)(\Delta_{h\varphi}^k \Phi_{x_i})(y)||(fu)(x_i)|\frac{\lambda_{m,i}^{\alpha,1-\alpha}}{u(x_i)} \le \mathcal{C}\|f\|_{C_u}\sum_{i=1}^{m}|u(y)(\Delta_{h\varphi}^k\Phi_{x_i})(y)|\frac{\lambda_{m,i}^{\alpha,1-\alpha}}{u(x_i)}, \tag{48}$$

with $k>0$, $h\le t$, t "small" and, then, taking the supremum on $y\in I_{h,k}$ and the supremum on $0<h\le t$, we get

$$\Omega_\varphi^k(\bar{K}_m f,t)_u \le \mathcal{C}\|f\|_{C_u}\sum_{i=1}^{m}\Omega_\varphi^k(\Phi_{x_i},t)_u\frac{\lambda_{m,i}^{\alpha,1-\alpha}}{u(x_i)} \le \mathcal{C}\|f\|_{C_u}\sup_{|x|\le 1}\Omega_\varphi^k(\Phi_x,t)_u\sum_{i=1}^{m}\frac{\lambda_{m,i}^{\alpha,1-\alpha}}{u(x_i)}.$$

Therefore, by (35), (20), (23) and (45), we obtain

$$E_m(\bar{K}_m f)_u \le \frac{\mathcal{C}}{m^r}\|f\|_{C_u},\quad \mathcal{C}\ne\mathcal{C}(m,f). \tag{49}$$

Now replacing f by $\bar{K}_m f$ into (47) we get

$$\|(\bar{K}-\bar{K}_m)\bar{K}_m f\|_{C_u} \le \frac{\mathcal{C}}{m^r}\|\bar{K}_m f\|_{C_u} + \mathcal{C}E_{m-1}(\bar{K}_m f)_u. \tag{50}$$

Hence from (42) and (49), (44) follows.

Now, since all the hypotheses of [22, Theorem 4.1.1, p.106] are fulfilled, we deduce that, for a sufficiently large m, the inverse operators $(I+\mu\bar{K}_m)^{-1}$ exist and are uniformly bounded with respect to m and, moreover,

$$\|(f^*-f_m^*)u\| \sim \|(\bar{K}-\bar{K}_m)f^*\|_{C_u}.$$

But, since under our assumptions both G and $\bar{K}f$ belong to $Z_r(u)$, we get that the unique solution f^* of (14) belongs to $Z_r(u)$, too, and then, by (47) and (36), we obtain (26).

Finally, proceeding as in [22, pp. 112–113], it is possible to prove that

$$\|A_m\| \le \|I + \mu \bar{K}_m\|_{C_u \to C_u} \tag{51}$$

and

$$\|A_m^{-1}\| \le \|(I + \mu \bar{K}_m)^{-1}\|_{C_u \to C_u} \le \mathcal{C}, \quad \mathcal{C} \ne \mathcal{C}(m), \tag{52}$$

where the matrix norm is the uniform one. Consequently by (51) and (52), (25) follows. □

In order to prove Proposition 4.1 we need some preliminary results.

Proposition 5.2. *If $0 < \alpha < 1$ and $f \in Z_r$, then we have*

$$\sup_{1 \le i \le m} v^{\alpha,1-\alpha}(x_i) \left| (\widehat{D} f)(x_i) - (\widehat{D} L_{m+1}^{-\alpha,\alpha-1} f)(x_i) \right| \le \mathcal{C} \frac{\log m}{m^r} \|f\|_{Z_r}, \tag{53}$$

where $\mathcal{C} \ne \mathcal{C}(m, k, f)$.

In order to prove Proposition 5.2 we will use the following proposition.

Proposition 5.3. *Let $0 < \alpha < 1, f \in C^0([-1, 1])$ and let $P_m \in \mathbb{P}_m$ be a polynomial such that*

$$\|f - P_m\| \le c E_m(f)$$

holds with some positive constant $c \ne c(m, f)$. Then

$$\inf_{P \in \mathbb{P}_m} \|v^{\alpha,1-\alpha} \widehat{D}(f - P_m)\| \le \mathcal{C} \left[E_m(f) \log m + \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f, t)}{t} \mathrm{d}t \right], \tag{54}$$

where $1 \le k < m$ and $\mathcal{C} \ne \mathcal{C}(m, f)$.

Proof. Using the following inequality proved in [24, Theorem 2.1]

$$\left| v^{\alpha,1-\alpha}(y) \int_{-1}^{1} \frac{f(x)}{x - y} v^{-\alpha,\alpha-1}(x) \mathrm{d}x \right| \le \mathcal{C} \left[\|f\| + \int_0^1 \frac{\omega_\varphi(f, t)}{t} \mathrm{d}t \right], \quad \mathcal{C} \ne \mathcal{C}(m, f),$$

we get

$$|v^{\alpha,1-\alpha}(y)(\widehat{D} f)(y)| \le \mathcal{C} \left[\|f\| + \int_0^1 \frac{\omega_\varphi(f, t)}{t} \mathrm{d}t \right].$$

Then we have

$$\|v^{\alpha,1-\alpha} \widehat{D}(f - P_m)\| \le \mathcal{C} \left[E_m(f) + \int_0^1 \frac{\omega_\varphi(f - P_m, t)}{t} \mathrm{d}t \right].$$

Moreover

$$\int_0^1 \frac{\omega_\varphi(f - P_m, t)}{t} \mathrm{d}t = \int_0^{\frac{1}{m}} \frac{\omega_\varphi(f - P_m, t)}{t} \mathrm{d}t + \int_{\frac{1}{m}}^1 \frac{\omega_\varphi(f - P_m, t)}{t} \mathrm{d}t.$$

Taking into account that [15, Lemma 2.1]

$$\int_0^{\frac{1}{m}} \frac{\omega_\varphi(f - P_m, t)}{t} \mathrm{d}t \le \mathcal{C} \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f, t)}{t} \mathrm{d}t, \quad k < m,$$

and that

$$\int_{\frac{1}{m}}^1 \frac{\omega_\varphi(f - P_m, t)}{t} \mathrm{d}t \le \mathcal{C} E_m(f) \log m,$$

we conclude

$$\int_0^1 \frac{\omega_\varphi(f - P_m, t)}{t} \mathrm{d}t \le \mathcal{C} \left[E_m(f) \log m + \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f, t)}{t} \mathrm{d}t \right]$$

and then (54) follows. □

Proof of Proposition 5.2. Denoting by P_m the polynomial of best approximation of degree m of f in the space $C^0([-1,1])$, we get

$$\begin{aligned}|(\widehat{D}f)(y)-(\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(y)| &= |\widehat{D}(f-L_{m+1}^{-\alpha,\alpha-1}f)(y)|\\ &\le |\widehat{D}(f-P_m)(y)|+|\widehat{D}L_{m+1}^{-\alpha,\alpha-1}(f-P_m)(y)|,\end{aligned}$$

and, therefore, using (28),

$$\begin{aligned}v^{\alpha,1-\alpha}(x_i)|(\widehat{D}f)(x_i)-(\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(x_i)| &\le v^{\alpha,1-\alpha}(x_i)|\widehat{D}(f-P_m)(x_i)|+\frac{1}{\pi}v^{\alpha,1-\alpha}(x_i)\left|\sum_{j=1}^{m+1}\frac{f(t_j)-P_m(t_j)}{t_j-x_i}\lambda_{m+1,j}^{-\alpha,\alpha-1}\right|\\ &=: B_1+B_2. \qquad (55)\end{aligned}$$

Using (54) and (36) we obtain

$$B_1\le \mathcal{C}\frac{\log m}{m^r}\|f\|_{Z_r}. \qquad (56)$$

Recalling that (see [25]) $\lambda_{m+1,j}^{-\alpha,\alpha-1}\sim \Delta t_j\, v^{-\alpha,\alpha-1}(t_j)$, $\Delta t_j=t_{j+1}-t_j$, $t_{m+2}=1$, we obtain

$$B_2\le \mathcal{C}\|f-P_m\|v^{\alpha,1-\alpha}(x_i)\sum_{j=1}^{m+1}\frac{\Delta t_j}{|x_i-t_j|}v^{-\alpha,\alpha-1}(t_j).$$

Moreover, since in virtue of (29), we have (see for instance [24, (5.16)])

$$\sum_{j=1}^{m+1}\frac{\Delta t_j}{|x_i-t_j|}v^{-\alpha,\alpha-1}(t_j)\le \mathcal{C}v^{-\alpha,\alpha-1}(x_i)\log m,$$

we get

$$B_2\le \mathcal{C}(\log m)E_{m-1}(f)$$

and, applying (36), we obtain

$$B_2\le \mathcal{C}\frac{\log m}{m^r}\|f\|_{Z_r}. \qquad (57)$$

Combining (55) with (56) and (57), (53) follows. □

In [26] the following lemma is proved.

Lemma 5.1. *For ρ,θ,γ and δ satisfying*

$$\begin{aligned}&\max\left\{0,\frac{\rho}{2}+\frac{1}{4}\right\}\le\gamma<\min\left\{\frac{\rho}{2}+\frac{3}{4},1+\rho\right\}\\ &\max\left\{0,\frac{\theta}{2}+\frac{1}{4}\right\}\le\delta<\min\left\{\frac{\theta}{2}+\frac{3}{4},1+\theta\right\}\end{aligned} \qquad (58)$$

and, for every $f\in C_{v^{\gamma,\delta}}$, we have

$$\|(L_m^{\rho,\theta}f)v^{\gamma,\delta}\|\le \mathcal{C}\log m\|fv^{\gamma,\delta}\| \qquad (59)$$

or, equivalently,

$$\|(f-L_m^{\rho,\theta}f)v^{\gamma,\delta}\|\le \mathcal{C}\log mE_{m-1}(f)_{v^{\gamma,\delta}}, \qquad (60)$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$.

Proposition 5.4. *If $0<\alpha<1$ and γ,δ satisfy (33), then for $f\in Z_r$ we have*

$$\|(\widehat{D}f-\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)v^{\alpha+\gamma,1-\alpha+\delta}\|\le \mathcal{C}\frac{\log^2 m}{m^r}\|f\|_{Z_r}, \qquad (61)$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$.

Proof. In virtue of (9), one has

$$(\widehat{D}L_{m+1}^{-\alpha,\alpha-1})f(y)=\sum_{j=1}^{m+1}f(t_j)(\widehat{D}l_{m+1,j}^{-\alpha,\alpha-1})(y)\in\mathbb{P}_{m-1}$$

and, then,

$$\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f=L_m^{\alpha,1-\alpha}\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f.$$

In order to prove estimate (61) we observe that, for any $y\in[-1,1]$, we can write

$$\begin{aligned}|(\widehat{D}f)(y)-(\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(y)|v^{\alpha+\gamma,1-\alpha+\delta}(y) &= |(\widehat{D}f)(y)-(L_m^{\alpha,1-\alpha}\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(y)|v^{\alpha+\gamma,1-\alpha+\delta}(y)\\ &\le |(\widehat{D}f)(y)-(L_m^{\alpha,1-\alpha}\widehat{D}f)(y)|v^{\alpha+\gamma,1-\alpha+\delta}(y)\\ &\quad+|(L_m^{\alpha,1-\alpha}\widehat{D}f)(y)-(L_m^{\alpha,1-\alpha}\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(y)|v^{\alpha+\gamma,1-\alpha+\delta}(y)\\ &=: A_1+A_2.\end{aligned}$$

Applying (60) and (36) to the function $\widehat{D}f$ and taking into account (19), we have

$$\begin{aligned}A_1 &\le \mathcal{C}\log mE_{m-1}(\widehat{D}f)_{v^{\alpha+\gamma,1-\alpha+\delta}}\\ &\le \mathcal{C}\frac{\log m}{m^r}\|\widehat{D}f\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}\\ &\le \mathcal{C}\frac{\log m}{m^r}\|f\|_{Z_r}.\end{aligned}\tag{62}$$

Now let us estimate A_2. For $\gamma,\delta\ge 0$, we obtain

$$\begin{aligned}|L_m^{\alpha,1-\alpha}(\widehat{D}f-\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(y)| &\le \sum_{i=1}^m v^{\alpha,1-\alpha}(x_i)|(\widehat{D}f)(x_i)-(\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(x_i)|\frac{|l_{m,i}^{\alpha,1-\alpha}(y)|}{v^{\alpha+\gamma,1-\alpha+\delta}(x_i)}v^{\gamma,\delta}(x_i)\\ &\le \mathcal{C}\sup_{1\le i\le m}v^{\alpha,1-\alpha}(x_i)|(\widehat{D}f)(x_i)-(\widehat{D}L_{m+1}^{-\alpha,\alpha-1}f)(x_i)|\sum_{i=1}^m\frac{|l_{m,i}^{\alpha,1-\alpha}(y)|}{v^{\alpha+\gamma,1-\alpha+\delta}(x_i)}.\end{aligned}$$

Then, by (53) and (59), we get

$$\begin{aligned}A_2 &\le \mathcal{C}\frac{\log m}{m^r}\|f\|_{Z_r}\|v^{\gamma,\delta}\|\max_{|y|\le 1}v^{\alpha+\gamma,1-\alpha+\delta}(y)\sum_{i=1}^m\frac{|l_{m,i}^{\alpha,1-\alpha}(y)|}{v^{\alpha+\gamma,1-\alpha+\delta}(x_i)}\\ &= \mathcal{C}\frac{\log m}{m^r}\|f\|_{Z_r}\|L_m^{\alpha,1-\alpha}\|_{C_{v^{\alpha+\gamma,1-\alpha+\delta}}\to C_{v^{\alpha+\gamma,1-\alpha+\delta}}}\\ &\le \mathcal{C}\frac{\log^2 m}{m^r}\|f\|_{Z_r}.\end{aligned}\tag{63}$$

From (62) and (63) the thesis follows. □

Proof of Proposition 4.1. Denote by A_m and A_m^* the matrices of the coefficients of the linear systems (17) and (30), respectively. In the proof of Theorem 3.1 it was shown that, for $m>m_0$, the inverse matrix A_m^{-1} exists and (52) holds true. Since

$$A_m^*=A_m[I+A_m^{-1}(A_m^*-A_m)],$$

by Neumann series theorem, if

$$\lim_m\|A_m^{-1}(A_m^*-A_m)\|=0\tag{64}$$

then, for a sufficiently large m, $(A_m^*)^{-1}$ exists. In order to prove (64), in virtue of (52), we have only to show that

$$\lim_m\|A_m^*-A_m\|=0.$$

By using (53), we get

$$\begin{aligned}\|A_m^*-A_m\| &= |\mu|\max_{1\le i\le m}\sum_{k=1}^m\left|\lambda_{m,k}^{\alpha,1-\alpha}\frac{v^{\alpha+\gamma,1-\alpha+\delta}(x_i)}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)}[\Phi_{m+1}(x_k,x_i)-\Phi(x_k,x_i)]\right|\\ &\le \mathcal{C}\frac{\log m}{m^r}\sup_{|x|\le 1}\|k_x\|_{Z_r}\|v^{\gamma,\delta}\|\sum_{k=1}^m\frac{\lambda_{m,k}^{\alpha,1-\alpha}}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)}.\end{aligned}$$

Thus from (15), (45) and (23), it follows

$$\|A_m^* - A_m\| \le \mathcal{C}\frac{\log m}{m^r}, \quad \mathcal{C} \neq \mathcal{C}(m). \tag{65}$$

Then, for a sufficiently large m, $(A_m^*)^{-1}$ exists and

$$\begin{aligned}\|(A_m^*)^{-1}\| &\le \|[I + A_m^{-1}(A_m^* - A_m)]^{-1}\|\|A_m^{-1}\| \\ &\le \frac{\|A_m^{-1}\|}{1 - \|A_m^{-1}\|\|A_m^* - A_m\|}\end{aligned}$$

from which, taking into account (65), we deduce

$$\lim_m \|(A_m^*)^{-1}\| \le \lim_m \|A_m^{-1}\|. \tag{66}$$

On the other hand, since

$$\|A_m^*\| \le \|A_m\|[\|I\| + \|A_m^{-1}\|\|A_m^* - A_m\|],$$

one also has

$$\lim_m \|A_m^*\| \le \lim_m \|A_m\| \tag{67}$$

and, then, combining (66) and (67), one can deduce

$$\lim_m \frac{\mathrm{cond}(A_m^*)}{\mathrm{cond}(A_m)} \le 1.$$

Since, using

$$A_m = A_m^*[I + (A_m^*)^{-1}(A_m - A_m^*)],$$

in an analogous way, one can also prove

$$\lim_m \frac{\mathrm{cond}(A_m)}{\mathrm{cond}(A_m^*)} \le 1,$$

(32) follows.

Now we prove (34). Since we can write

$$\|(f^* - f_m^{**})v^{\alpha+\gamma,1-\alpha+\delta}\| \le \|(f^* - f_m^*)v^{\alpha+\gamma,1-\alpha+\delta}\| + \|(f_m^* - f_m^{**})v^{\alpha+\gamma,1-\alpha+\delta}\|$$

and (26) holds true if f_m^* is the unique solution of Eq. (16), we have only to estimate $\|(f_m^* - f_m^{**})v^{\alpha+\gamma,1-\alpha+\delta}\|$.

We can write

$$\begin{aligned}|(f_m^* - f_m^{**})(y)|v^{\alpha+\gamma,1-\alpha+\delta}(y) &\le |(G - G_{m+1})(y)|v^{\alpha+\gamma,1-\alpha+\delta}(y) \\ &\quad + \left|\mu\sum_{k=1}^m \left[\Phi(x_k,y)a_k^* - \Phi_{m+1}(x_k,y)\bar{a}_k^*\right]\frac{\lambda_{m,k}^{\alpha,1-\alpha}}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)}\right| v^{\alpha+\gamma,1-\alpha+\delta}(y) \\ &=: A + B.\end{aligned}$$

Recalling the definitions of G and G_{m+1}, by (61), we deduce

$$A \le \mathcal{C}\frac{\log^2 m}{m^r}\|g\|_{Z_r}$$

whereas, setting $\mathbf{a} = (a_k^*)_{k=1,n}$ and $\bar{\mathbf{a}} = (\bar{a}_k^*)_{k=1,n}$ and taking into account (45), under assumption (15), we have

$$\begin{aligned}B &\le |\mu|\left|\sum_{k=1}^m \Phi(x_k,y)(a_k^* - \bar{a}_k^*)\frac{\lambda_{m,k}^{\alpha,1-\alpha}}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)}\right| v^{\alpha+\gamma,1-\alpha+\delta}(y) \\ &\quad + |\mu|\left|\sum_{k=1}^m [\Phi(x_k,y) - \Phi_{m+1}(x_k,y)]\bar{a}_k^*\frac{\lambda_{m,k}^{\alpha,1-\alpha}}{v^{\alpha+\gamma,1-\alpha+\delta}(x_k)}\right| v^{\alpha+\gamma,1-\alpha+\delta}(y) \\ &\le \mathcal{C}\sup_{|y|\le 1} v^{\alpha+\gamma,1-\alpha+\delta}(y)\|\Phi_y\|\|\mathbf{a} - \bar{\mathbf{a}}\|_\infty \\ &\quad + \mathcal{C}\sup_{|y|\le 1} v^{\alpha+\gamma,1-\alpha+\delta}(y)\sup_{1\le k\le m}|\Phi(x_k,y) - \Phi_{m+1}(x_k,y)|\|\bar{\mathbf{a}}\|_\infty =: B_1 + B_2.\end{aligned} \tag{68}$$

In order to estimate B_1, by (21) and (24), it is sufficient to estimate $\|\mathbf{a}-\bar{\mathbf{a}}\|_\infty$. Since $\mathbf{a}$ and $\bar{\mathbf{a}}$ are the solutions of the systems

$$A_m\mathbf{a}=\mathbf{b},\quad \mathbf{b}=\left(G(x_i)v^{\alpha+\gamma,1-\alpha+\delta}(x_i)\right)_{i=1,m},$$
$$A_m^*\bar{\mathbf{a}}=\mathbf{b}^*,\quad \mathbf{b}^*=\left(G_{m+1}(x_i)v^{\alpha+\gamma,1-\alpha+\delta}(x_i)\right)_{i=1,m},$$

respectively, the following estimate holds true (see, for instance, [27])

$$\|\mathbf{a}-\bar{\mathbf{a}}\|_\infty\le\frac{\|A_m^{-1}\|}{1-\|A_m^{-1}(A_m^*-A_m)\|}\left(\|\mathbf{b}-\mathbf{b}^*\|_\infty+\|A_m-A_m^*\|\|\mathbf{a}\|_\infty\right).$$

Now, let us observe that by (53), (22) and (15), it follows

$$\begin{aligned}\|\mathbf{b}-\mathbf{b}^*\|_\infty &= \max_{1\le i\le m} v^{\alpha,1-\alpha}(x_i)|G(x_i)-G_m(x_i)|v^{\gamma,\delta}(x_i)\\ &\le \mathcal{C}\frac{\log m}{m^r}\|g\|_{Z_r},\end{aligned}\tag{69}$$

whereas, for a sufficiently large m, in virtue of (26), one has

$$\begin{aligned}\|\mathbf{a}\|_\infty &= \max_{1\le k\le n}|f_m^*(x_k)v^{\alpha+\gamma,1-\alpha+\delta}(x_k)|\\ &\le \|(f^*-f_m^*)v^{\alpha+\gamma,1-\alpha+\delta}\|+\|f^*v^{\alpha+\gamma,1-\alpha+\delta}\|\\ &\le \frac{\mathcal{C}}{m^r}\|f^*\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}+\|f^*v^{\alpha+\gamma,1-\alpha+\delta}\|\\ &\le \mathcal{C}\|f^*\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})},\end{aligned}\tag{70}$$

being $f^*\in Z_r(v^{\alpha+\gamma,1-\alpha+\delta})$.

Then, by (52) and (65), (69) and (70), for a sufficiently large m, we can deduce that

$$B_1\le\mathcal{C}\|\mathbf{a}-\bar{\mathbf{a}}\|_\infty\le\mathcal{C}\frac{\log m}{m^r}\left(\|g\|_{Z_r}+\|f^*\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}\right)\tag{71}$$

with $\mathcal{C}\ne\mathcal{C}(m)$.

It remains to estimate B_2. Recalling the definitions of Φ and Φ_{m+1}, by (61), for any $k=1,\dots,m$, we can write

$$B_2\le\mathcal{C}\frac{\log^2 m}{m^r}\sup_{1\le k\le m}\|k_{x_k}\|_{Z_r}\|\bar{\mathbf{a}}\|_\infty.$$

Since, by (70) and (71), for a sufficiently large m, we have,

$$\begin{aligned}\|\bar{\mathbf{a}}\|_\infty &\le \|\mathbf{a}-\bar{\mathbf{a}}\|_\infty+\|\mathbf{a}\|_\infty\\ &\le \mathcal{C}\frac{\log m}{m^r}\left(\|g\|_{Z_r}+\|f^*\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}\right)+\mathcal{C}\|f^*\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}\\ &\le \mathcal{C}\left(\|g\|_{Z_r}+\|f^*\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}\right),\end{aligned}$$

with $\mathcal{C}\ne\mathcal{C}(m)$, by (23) we deduce

$$B_2\le\mathcal{C}\frac{\log^2 m}{m^r}.\tag{72}$$

Combining (71) and (72) with (68), we get

$$B\le\mathcal{C}\frac{\log^2 m}{m^r}.$$

From the estimates obtained for A and B (34) follows. □

6. Numerical examples

In this section, we show by some examples that our theoretical results are confirmed by the numerical tests.

Since we do not know the exact solutions of the integral equations, we will think as exact their approximate solutions

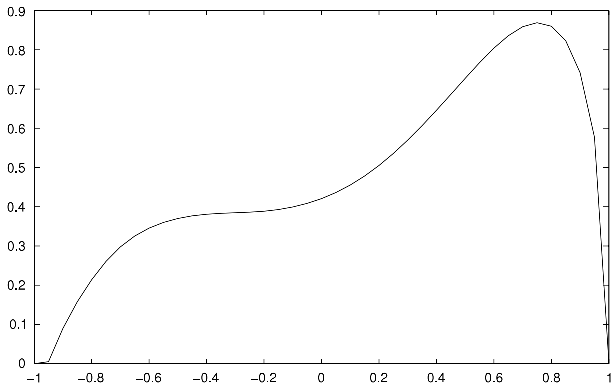

Fig. 1. f_{16}^{**}.

Table 1
$f_m^{**}(y)$.

m	$y = 0.5$	$y = 0.9$	cond
8	0.56102654811068	0.175586	1.083174230238684
16	0.56102654811068	0.17558686512257	1.092186158096304

Table 2
$f_m^{**}(y)$.

m	$y = 0.5$	$y = 0.3$	cond
8	0.3556	0.29	1.102100510518493
16	0.3556	0.2986	1.108232288445907
32	0.355628	0.2986	1.109985609555607
64	0.355628	0.2986896	1.110455512559596
128	0.35562839481617	0.2986896751	1.110577243393594

obtained for $m = 512$, and in all the tables we will report only the digits which are correct according to them. Moreover, in every table we will denote by cond the condition number of the matrix of the solved linear system.

Example 1. We consider the following integral equation

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}\sqrt{1-x^2}+\frac{1}{3}\int_{-1}^{1}\arctan(y^2+1)(x^3+x^5)\sqrt{1-x^2}\mathrm{d}x = y\cos y. \tag{73}$$

The kernel $k(x, y) = \arctan(y^2 + 1)(x^3 + x^5)$ and the right-hand side $g(y) = y\cos y$ satisfy (13) and, according to (15), we choose $\gamma = \delta = 0$. Since the regularized equation of (73) has a unique solution in $C^0([-1, 1])$ (the norm of its integral operator is less than 1), (73) has a unique solution in $C^0([-1, 1])$, too. Solving system (30) and using (31), we compute the approximate solution f_m^{**}. Since both the kernel and the known term are analytic functions, the convergence is very fast (see Table 1).

In Fig. 1 we show the graph of the function f_{16}^{**}. The max absolute error is of the order of 10^{-15}.

Example 2. Let us consider the following integral equation

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}\sqrt{1-x^2}+\frac{1}{3}\int_{-1}^{1}\sin yx^5\sqrt{1-x^2}\mathrm{d}x = y|y|^3\cos y. \tag{74}$$

The kernel $k(x, y) = \sin yx^5$ and the right-hand side $g(y) = y|y|^3\cos y$ satisfy (13) and in virtue of (15) we choose $\gamma = \delta = 0$. The regularized equation of (74) has a unique solution in $C^0([-1, 1])$ (the norm of its integral operator is less than 1) and, then, (74) has a unique solution in $C^0([-1, 1])$, too. Thus, solving system (30) and using (31), we compute the approximate solution f_m^{**} having rate of convergence of the order of $\frac{\log^2 m}{m^3}$ (see Table 2).

In Fig. 2 is given the graph of the function f_{128}^{**}. The max absolute error is of the order of 10^{-11}.

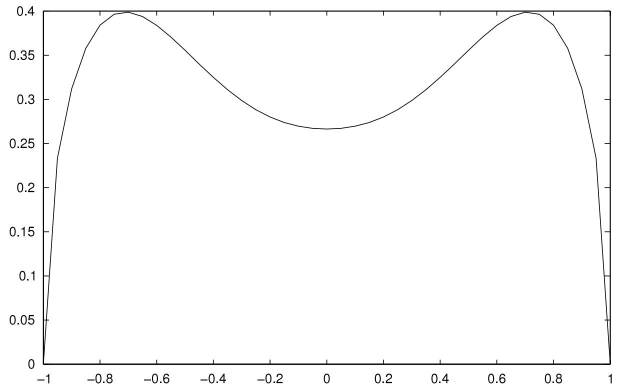

Fig. 2. f_{128}^{**}.

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